

# A note on the main theorem for absolutely monotonic functions

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## Abstract

We prove that the main theorem for absolutely monotonic functions on  $(0, \infty)$  from the book Mitrinović D.S., Pečarić J.E., Fink A.M. "Classical And New Inequalities In Analysis", Kluwer Academic Publishers, 1993, is not valid.

In the classical book [1], chapter XIII, page 365, there is a definition of absolutely monotonic on  $(0, \infty)$  functions.

**Definition.** A function  $f(x)$  is said to be *absolutely monotonic on*  $(0, \infty)$  if it has derivatives of all orders and

$$f^{(k)}(x) \geq 0, x \in (0, \infty), k = 0, 1, 2, \dots \quad (1)$$

For absolutely monotonic functions the next integral representation is essential:

$$f(x) = \int_0^\infty e^{xt} d\sigma(t), \quad (2)$$

where  $\sigma(t)$  is bounded and nondecreasing and the integral converges for all  $x \in (0, \infty)$ .

Also the basic set of inequalities is considered.

Let  $f(x)$  be an absolutely monotonic function on  $(0, \infty)$ . Then

$$f^{(k)}(x)f^{(k+2)}(x) \geq \left(f^{(k+1)}(x)\right)^2, k = 0, 1, 2, \dots \quad (3)$$

After this definition in the book [1] the result which we classify as the main theorem for absolutely monotonic functions on  $(0, \infty)$  is formulated (theorem 1, page 366).

## The main theorem for absolutely monotonic functions.

*The above definition, integral representation and basic set of inequalities are equivalent.*

It means:

$$(1) \Leftrightarrow (2) \Leftrightarrow (3). \quad (4)$$

In the book [1] for  $(1) \Leftrightarrow (2)$  the reference is given to [2], and an equivalence  $(2) \Leftrightarrow (3)$  is proved, it is in fact a consequence of Chebyshev inequality.

In this notes we consider a counterexample to the equivalence  $(1) \Leftrightarrow (2)$  of Widder. So unfortunately it seems that the main theorem for absolutely monotonic functions in the book [1] **is not valid !!!**.

This counterexample is very simple so it is strange enough it was not found before.

Really, consider a function  $f(x) = x^2 + 1$ . Obviously for all  $x \in (0, \infty)$

$$f(x) \geq 0, f'(x) = 2x \geq 0, f''(x) = 2 \geq 0, f^{(k)}(x) = 0 \geq 0, k > 2. \quad (5)$$

So this function  $f(x)$  is in the class of absolutely monotonic functions on  $(0, \infty)$  due to the definition (1). If  $(1) \Rightarrow (3)$  is valid then the next inequality must be true as a special case of (3) for all  $x \in (0, \infty)$

$$f(x)f''(x) \geq (f'(x))^2 \Leftrightarrow 2(x^2 + 1) \geq 4x^2 \Leftrightarrow 1 \geq x^2$$

but this is not valid for all  $x \in (0, \infty)$ .

As a conclusion we see that implication  $(1) \Rightarrow (3)$  in [1] is not valid. It also means that implication  $(1) \Rightarrow (2)$  is also not valid. The implication  $(2) \Rightarrow (3)$  is obviously valid due to the Chebyshev inequality.

And consequently also the theorem 2 in [1], pages 366–367 on determinantal inequalities is not valid too if based only on definition (1).

In some papers the above implications are used to derive new results for absolutely monotonic functions. It seems not to be valid. One way is to change the main theorem on absolutely monotonic functions to a proper one, otherwise for all special cases an integral representation must be proved independently.

Comment 1. On the other hand everything is OK with theorems on completely monotonic functions. An integral representation for them in [3] include the additional condition

$$\lim_{x \rightarrow \infty} f(x) = 0.$$

May be something like it is needed also for absolutely monotonic functions.

Different aspects of completely monotonic functions are considered in ([2]–[3]), and also for example in the classical expository articles ([4]–[6]).

Comment 2. There are many ways to generalize notions of absolutely and completely monotonic functions. Among them all very interesting generalizations were proposed by Bulgarian mathematicians Nikola Obreshkov and Jaroslav Tagamlitskii.

Comment 3. With absolute and complete monotonicity different functional classes are deeply connected: Stieltjes, Pick, Bernstein, Schoenberg, Schur and others, cf. [7]. Just mention two recent papers [8] and [9].

The author is thankful to Prof. Ivan Dimovski for useful information and papers of N. Obreshkov and J. Tagamlitskii.

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